Fig. 2. In marked contrast to the uniformly good results obtained for skin friction, the refined integral method predicts a displacement thickness distribution that diverges drastically from the rapid rise of the exact solution when the skin friction becomes small (incipient blow-off), tending in fact toward the linear behavior given by the ordinary Kârmân-Pohlhausen and Rayleigh analogy approximations. Evidently, δ^* is quite sensitive to the skin-friction behavior near blow-off. It is also noteworthy from Fig. 2 that the local similarity approximation once again yields qualitatively correct results. Clearly, then, in spite of its generally good accuracy in predicting skin friction, the refined moment integral approach still possesses some deficiencies that would limit its usefulness in treating strong blowing and incipient separation problems where an accurate account of the displacement effects is essential.

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Reply by Author to G. R. Inger

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THE author wishes to thank Professor Inger for taking interest in the author's work.[1] His enthusiastic efforts in exploring the potential of the approximate method, currently being investigated by the author, are particularly appreciated. Stimulated by Inger's comments, the author feels that some further remarks about Ref. 1 are in order.

The nature and basic ideas of this approximate method along with its principal merits have been elucidated on several occasions in the *AIAA Journal* [see Ref. (2) in particular], and, therefore, require no further elaboration.

The merits of the present method are highlighted in Fig. 1 of the Comment which illustrates an interesting comparison on the results produced by a wide variety of approximate methods of different nature. In response to Inger's remarks on the obvious inadequacy of the method near blow-off, it is perhaps appropriate to reiterate here that the present method is only meant to be a refinement of the usual Kârmân-Pohlhausen (K-P) method. Being aware of the approximate nature of the method the present author certainly makes no claim as to its perfection. Obviously, Ref. 1 presented the method as a simple, practical tool for studying usual boundary-layer flows with surface mass transfer. In this form, it was never intended for studying the delicate and difficult problem of blow-off, separation or separated flows.

In light of the aforementioned discussion, Fig. 2 of the Comment provides further evidence to the accuracy of the present method. It is emphasized that due care must be exercised in observing the obvious region of its intended applications, i.e., $\varepsilon^2 Re_x < (\varepsilon^2 Re_x)_{\tau_w=0}$ (≈ 0.4 for the particular profile under discussion; $\varepsilon \equiv v_w/u_\infty$). Accordingly, Inger's calculation of δ^*, based on the solutions of Ref. (1), should have been limited to this region where the refined integral method [Ref. (1)] predicts positive skin friction. The importance of the effective displacement thickness in studying the problem of viscous-inviscid interaction in the presence of surface mass transfer has received considerable attention from most fluid dynamicists, including the present author[3] (unsteady, weak interactions). Interested readers are referred to Fannelop[4] and Li[5] among others, for a more thorough discussion on the subject and pertinent references. We only note that the conventional displacement thickness, δ^*, alone generally does not represent the total displacement effect. Modifications should be made to account for the effects of surface mass transfer.

In closing, the author is of the opinion that the merits of the refined K-P method, as illustrated and reiterated in Refs. 1 and 2, are best exploited when applications are made to the calculations of surface properties. The extension of the method to study problems of massive blowing, viscous interaction, or separated flows must proceed with caution.

Finally, the author would like to take this opportunity to correct a minor typographical error in Ref. 1. The suction strength in the caption of Fig. 4 should read $\varepsilon A^{1/2} = -1/(2)^{1/2}$.

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Comment on "Finite Elements for Axisymmetric Solids under Arbitrary Loadings with Nodes on Origin"

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IN a Technical Note, Belytschko[1] outlines how a finite element analysis of axisymmetric solids using linear displacement triangular ring elements may correctly include nodes lying on the

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axis of revolution. If arbitrary loading is represented by Fourier series, additional constraint conditions for the degrees of freedom of the axis nodes are derived from the consideration that, for any axis point, the strains associated with the assumed linear displacement field should not become singular.

It is the purpose of this Comment to point out that these constraint conditions follow directly from the general kinematical compatibility requirements, i.e., from the condition that the displacement field be continuous. Corresponding constraint conditions may be applied[2,3] to the analysis of axisymmetric solids using not only linear displacement but also quadratic and cubic displacement triangular ring elements.

Furthermore, if instead of the third of Belytschko's Eq. (3)[1] we use

$$u_\theta(r, z, \theta) = \sum_n u_\theta{}^n(r, z)(-\sin n\theta) + \sum_n u_\theta{}^{-n}(r, z) \cos n\theta \qquad (1)$$

the element stiffnesses and associated matrices will be the same for the symmetric and antisymmetric components of each harmonic, i.e., for n and $-n$ (with the exception of the zeroth harmonic). Thus the effort to set up the stiffness matrices is considerably reduced.

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Reply by Author to K. E. Buck

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REGARDLESS of whether the constraint conditions, Eqs. (5) are derived from continuity or the boundedness of the strains, the resulting conditions must be directly imposed on the displacement fields of elements with nodes on the origin to avoid singularities in the stiffness matrices. Reference 2 of the Comment does not mention this.

It should be added that maintaining the boundedness of all terms of the stiffness matrix is not essential in static analysis. If the singular terms are not omitted, the numerical integration of these terms then yields elements in the stiffness equations which are an order of magnitude larger than the others; these larger terms are essentially linear combinations of the constraints. However, this is a rather unsound manner for treating these terms and fails totally in the explicit integration of the transient equations. In that case, the stability limit on the time step is inversely proportional to the largest element in the stiffness matrix. Thus the inclusion of the unbounded stiffness matrix terms leads to prohibitively small stability limits and the explicit integration of the transient equations is impossible. Hence, the enforcement of the constraints on the element displacement field as given in Ref. 1 is mandatory for any application of the stiffness matrix to explicit transient solutions and also preferable in static solutions.

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Closed-Form Lift and Moment for Osborne's Unsteady Thin-Airfoil Theory

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IN Ref. 1 Osborne presented an approximate theory for the unsteady motion of a two-dimensional thin airfoil in subsonic flow. As applications, he wrote the lift and moment for an airfoil subject to three oscillating upwash distributions whose chordwise dependence can be expressed in a cosine series. In only one of these three cases was the lift and moment written in closed form. In the other two, they involved infinite series of products of Bessel functions.

One purpose of this Comment is to point out that all these series can be summed, so that closed form expressions for the lift and moment can be obtained in all the cases considered by Osborne, thereby simplifying considerably their use in numerical calculation. For example, these simplifications would have been of value in Osborne's recent discussion of compressibility effects in unsteady interactions between the blade rows of turbomachines.[2]

A second purpose is to present, also in closed form, the lift and moment for the case of pitching oscillations, to complement the plunging oscillation case given by Osborne.

The notation used will be that of Ref. 1, where the most general type of upwash considered has the form

$$v(x, t) = v_0 \exp(ivt - i\mu x/V) \qquad (1)$$

This is referred to by Osborne as "Kemp-type upwash" because it was first introduced in Ref. 3. In terms of the parameters

$$\lambda = \mu c/V, \quad \omega = vc/V, \quad \omega^i = \omega/\beta^2, \quad \lambda^i = M^2\omega^i, \quad \Lambda = \lambda + \lambda^i \qquad (2)$$

and the Bessel function abbreviations

$$J = J_0 - iJ_1, \quad C = K_1/(K_0 + K_1), \quad S(z) = [iz(K_0 + K_1)]^{-1} \qquad (3)$$

the lift and nose-up moment for upwash (1) is given by Osborne[1] in Eqs. (29) and (30) as

$$L(t) = 2\pi\beta^{-1}\rho_\infty cVv_0 e^{ivt}\Big\{J(\Lambda)[C(\omega^i)J(\lambda^i) + iJ_1(\lambda^i)] + i(\omega^i/\Lambda)[J_0(\lambda^i)J_1(\Lambda) - J_1(\lambda^i)J_0(\Lambda)] - 2i[(\Lambda - \omega^i)/\Lambda^2]\sum_{n=1}^{\infty} nJ_n(\lambda^i)J_n(\Lambda)\Big\} \qquad (4)$$

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